JOURNAL OF APPROXIMATION THEORY 7, 122–131 (1973)

L_1 Approximation of Vector-Valued Functions

M. P. CARROLL* AND H. W. MCLAUGHLIN

Rensselaer Polytechnic Institute, Troy, New York 12181

Communicated by Oved Shisha

Received June 8, 1970

INTRODUCTION

Let f_1 and f_2 denote real valued continuous functions defined on a compact Lebesgue measurable subset E of the real line. Let n denote a fixed positive integer and P the set of real polynomials of degree n or less. In [1] Dunham considered, for the case E a nondegenerate compact interval, the problem of minimizing the expression

$$\max\{\|f_1 - p\|, \|f_2 - p\|\} \qquad (p \in P),$$

where $\| \ \|$ denotes the supremum norm. In other notation this expression assumes the form

$$\big\| \, \|f_1 - p\|_{L_\infty}, \|f_2 - p\|_{L_\infty} \big\|_{l_\infty} \qquad (p \in P).$$

In this paper we study the corresponding problem of minimizing

$$\big\| \, \|f_1 - p\|_{L_1}, \|f_2 - p\|_{L_1} \big\|_{l_1} \qquad (p \in P);$$

i.e., minimizing

$$\int_E |f_1 - p| \, dx + \int_E |f_2 - p| \, dx \qquad (p \in P).$$

More generally, we consider the problem of minimizing

$$\sum_{i=1}^{m} \int_E |f_i - p| \, dx \qquad (p \in P),$$

where $f_1, ..., f_m$ are Lebesgue measurable real-valued functions on E.

* This paper is taken in part from a thesis by M. P. Carroll in partial fulfillment of the requirements for the Ph.D. degree in the Department of Mathematics at Rensselaer Polytechnic Institute.

In Section I we consider questions of existence and characterization of a polynomial of best approximation i.e., of a polynomial $q \in P$ such that

$$\sum_{i=1}^{m} \int_E | f_i - q | \, dx = \inf_{p \in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx.$$

The proofs, although fairly standard, are included for completeness.

In Section II we consider the question of uniqueness of a polynomial of best approximation. Our results indicate that when m is odd the polynomial which best approximates $(f_1, \ldots, f_m)$ is unique and when m is even, it may or may not be unique.

SECTION I

For E a nonempty compact Lebesgue measurable subset of the real line, $L^1(E)$ denotes the set of all Lebesgue integrable real-valued functions defined on E.

THEOREM 1. *Let m be a positive integer and let $f_i \in L^1(E)$ $(1 \leqslant i \leqslant m)$. Then there exists a polynomial $q \in P$ such that*

$$\sum_{i=1}^{m} \int_E | f_i - q | \, dx = \inf_{p \in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx.$$

Proof. Let $\langle p_k \rangle$ be a sequence in P such that

$$\lim_{k \to \infty} \sum_{i=1}^{m} \int_E | f_i - p_k | \, dx = \inf_{p \in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx.$$

For each k we have

$$\begin{aligned}
0 &\leqslant \int_E | p_k | \, dx \\
&\leqslant \int_E | f_1 - p_k | \, dx + \int_E | f_1 | \, dx \\
&\leqslant \sum_{i=1}^{m} \int_E | f_i - p_k | \, dx + \int_E | f_1 | \, dx \\
&= \inf_{p \in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx \\
&\quad + \left[\sum_{i=1}^{m} | f_i - p_k | \, dx - \inf_{p \in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx \right] + \int_E | f_1 | \, dx.
\end{aligned}$$

Since the term in brackets tends to zero as k tends to infinity the sequence $\langle \int_E | p_k | \, dx \rangle$ is uniformly bounded. Thus the sequence $\langle p_k \rangle$ contains a subsequence which converges, in the L^1 norm, to an element of P (see, e.g., [2, p. 16]). Without loss we assume that there exists an element $q \in P$ such that $\lim_{k\to\infty} \int_E | p_k - q | \, dx = 0$. Further, for each k

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{m} \int_E | f_i - q | \, dx - \inf_{p\in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx \\ &\leqslant \sum_{i=1}^{m} \int_E (| f_i - p_k | + | p_k - q |) \, dx - \inf_{p\in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx \\ &= m \int_E | p_k - q | \, dx + \left[\sum_{i=1}^{m} \int_E | f_i - p_k | \, dx - \inf_{p\in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx \right]. \end{aligned}$$

Since the term in brackets tends to zero as k approaches infinity, and since $\lim_{k\to\infty} \int_E | p_k - q | \, dx = 0$ we conclude that

$$\sum_{i=1}^{m} \int_E | f_i - q | \, dx = \inf_{p\in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx.$$

This completes the proof.

For a real valued function f, defined on E, denote the set of its zeros by Z_f.

THEOREM 2. *Let m be a positive integer and let $f_i \in L^1(E)$ $(1 \leqslant i \leqslant m)$. Then a polynomial $q \in P$ is a best approximant to $(f_1, ..., f_m)$, i.e.,*

$$\sum_{i=1}^{m} \int_E | f_i - q | \, dx = \inf_{p\in P} \sum_{i=1}^{m} \int_E | f_i - p | \, dx,$$

iff for every $p \in P$,

$$\sum_{i=1}^{m} \int_{Z_{f_i - q}} | p | \, dx \geqslant \left| \sum_{i=1}^{m} \int_E p \operatorname{sgn}(f_i - q) \, dx \right|. \tag{1}$$

Proof. Let q be a polynomial in P which best approximates $(f_1, ..., f_m)$ and suppose that for some $p_0 \in P$ (1) does not hold. Without loss we may assume that

$$\sum_{i=1}^{m} \int_{Z_{f_i - q}} | p_0 | \, dx < \sum_{i=1}^{m} \int_E p_0 \operatorname{sgn}(f_i - q) \, dx.$$

Now let t be a fixed positive real number and $M = \max_{x\in E} | p_0(x)|$, and define

$$E_{f_i} = \{x \in E: | f_i(x) - q(x)| \leqslant tM\} \qquad (1 \leqslant i \leqslant m).$$

Since $\operatorname{sgn}(f_i - q) = \operatorname{sgn}(f_i - q - tp_0)$ on $E \sim E_{f_i}$ $(1 \leqslant i \leqslant m)$, we have

$$\begin{aligned}
&\int_E |f_i - q - tp_0|\,dx \\
&\quad = \int_{E\sim E_{f_i}} (f_i - q - tp_0)\operatorname{sgn}(f_i - q)\,dx + \int_{E_{f_i}} |f_i - q - tp_0|\,dx \\
&\quad = \int_{E\sim E_{f_i}} |f_i - q|\,dx - \int_{E\sim E_{f_i}} tp_0 \operatorname{sgn}(f_i - q)\,dx + \int_{E_{f_i}} |f_i - q|\,dx \\
&\qquad - \int_{E_{f_i}} |f_i - q|\,dx + \int_{E_{f_i}} tp_0 \operatorname{sgn}(f_i - q)\,dx \\
&\qquad - \int_{E_{f_i}} tp_0 \operatorname{sgn}(f_i - q)\,dx + \int_{E_{f_i}} |f_i - q - tp_0|\,dx \\
&\quad = \int_E |f_i - q|\,dx - \int_E tp_0 \operatorname{sgn}(f_i - q)\,dx + \int_{E_{f_i}} (f_i - q - tp_0) \\
&\qquad \times [\operatorname{sgn}(f_i - q - tp_0) - \operatorname{sgn}(f_i - q)]\,dx \qquad (1 \leqslant i \leqslant m).
\end{aligned}$$

Since $|f_i - q - tp_0| \leqslant 2tM$ on E_{f_i} $(1 \leqslant i \leqslant m)$, we have

$$\begin{aligned}
&\int_E |f_i - q - tp_0|\,dx - \int_E |f_i - q|\,dx \\
&\quad \leqslant t\left[\int_{Z_{f_i-q}} |p_0|\,dx - \int_E p_0 \operatorname{sgn}(f_i - q)\,dx + 4Mx(E_{f_i} - Z_{f_i-q})\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{m} \int_E |f_i - q - tp_0|\,dx - \sum_{i=1}^{m} \int_E |f_i - q|\,dx \\
&\quad \leqslant t\left[\sum_{i=1}^{m} \int_{Z_{f_i-q}} |p_0|\,dx - \sum_{i=1}^{m} \int_E p_0 \operatorname{sgn}(f_i - q)\,dx \right. \\
&\qquad \left. + 4M \sum_{i=1}^{m} x(E_{f_i} \sim Z_{f_i-q})\right].
\end{aligned}$$

For each i $(1 \leqslant i \leqslant m)$ E_{f_i} is nondecreasing (in the sense of set inclusion) with respect to t; and since $xE < \infty$, $\lim_{t\to 0} x(E_{f_i} \sim Z_{f_i-q}) = 0\,(1 \leqslant i \leqslant m)$. Thus there exists $\bar{t} > 0$ such that

$$\sum_{i=1}^{m} \int_E |f_i - q - \bar{t}p_0|\,dx - \sum_{i=1}^{m} \int_E |f_i - q|\,dx < 0,$$

which contradicts the choice of q.

Conversely, let q be a polynomial in P for which (1) holds for every $p \in P$. Then for any $p \in P$ we have

$$\sum_{i=1}^{m} \int_E |f_i - q| \, dx$$

$$= \sum_{i=1}^{m} \int_E (f_i - p) \operatorname{sgn}(f_i - q) \, dx + \sum_{i=1}^{m} \int_E (p - q) \operatorname{sgn}(f_i - q) \, dx$$

$$= \sum_{i=1}^{m} \int_{E \sim Z_{f_i - q}} (f_i - p) \operatorname{sgn}(f_i - q) \, dx + \sum_{i=1}^{m} \int_E (p - q) \operatorname{sgn}(f_i - q) \, dx.$$

From this and (1) we obtain

$$\sum_{i=1}^{m} \int_E |f_i - q| \, dx$$

$$\leqslant \sum_{i=1}^{m} \int_{E \sim Z_{f_i - q}} (f_i - p) \operatorname{sgn}(f_i - q) \, dx + \sum_{i=1}^{m} \int_{Z_{f_i - q}} |p - q| \, dx.$$

Since $|p - q| = |f_i - p|$ on $Z_{f_i - q}$ $(1 \leqslant i \leqslant m)$ we obtain

$$\sum_{i=1}^{m} \int_E |f_i - q| \, dx \leqslant \sum_{i=1}^{m} \int_E |f_i - p| \, dx,$$

which completes the proof.

Section II

In this section we prove uniqueness theorems for the cases $m = 2$ and, $m = 3$, after which we state without proof two uniqueness theorems, one for m an even integer and one for m an odd integer. The proofs of these last two theorems parallel the proofs of the theorems for the cases $m = 2$ and $m = 3$.

We say that a point x_0 interior to a real interval I is a zero crossing of a real-valued function f defined on I if $f(x_0) = 0$ and $f(x) \cdot (x - x_0)$ has the same sign in some deleted neighborhood of x_0. (f may be identically zero in a neighborhood of x_0.)

Lemma 1. *Let I be a nondegenerate compact interval of the real line. Let f_1 and f_2 be continuous real-valued functions on I such that $f_1 \leqslant f_2$ on I. Let q be a polynomial in P such that*

$$\int_I |f_1 - q| \, dx + \int_I |f_2 - q| \, dx = \inf_{p \in P} \left[\int_I |f_1 - p| \, dx + \int_I |f_2 - p| \, dx \right].$$

If there exists $\bar{x} \in I$ *such that* $[f_1(\bar{x}) - q(\bar{x})][f_2(\bar{x}) - q(\bar{x})] > 0$ *and if* $f_i - q$ *has at most a finite number of zeros, then* $f_i - q$ *has more than* n *zeros on* I $(i = 1, 2)$.

Proof. Letting $p \equiv 1$ and $m = 2$ in Theorem 2 gives

$$0 \geqslant \left| \int_I \operatorname{sgn}(f_1 - q)\, dx + \int_I \operatorname{sgn}(f_2 - q)\, dx \right|.$$

Since $|\operatorname{sgn}(f_1 - q) + \operatorname{sgn}(f_2 - q)| = 2$ in a neighborhood of $\bar{x}$ it follows that both $f_1 - q$ and $f_2 - q$ have a zero crossing in I. Let $x_1 < x_2 < \cdots < x_k$ be the zero crossings of $f_1 - q$. Since $\operatorname{sgn}(f_1 - q) > 0$ implies that $\operatorname{sgn}(f_1 - q) + \operatorname{sgn}(f_2 - q) > 0$ and $\operatorname{sgn}(f_1 - q) < 0$ implies that

$$\operatorname{sgn}(f_1 - q) + \operatorname{sgn}(f_2 - q) \leqslant 0,$$

and since $|\operatorname{sgn}(f_1 - q) + \operatorname{sgn}(f_2 - q)| = 2$ in a neighborhood of $\bar{x}$, it follows that

$$\int_I \bar{p} \operatorname{sgn}(f_1 - q)\, dx + \int_I \bar{p} \operatorname{sgn}(f_2 - q)\, dx \neq 0$$

where

$$\bar{p}(x) = \prod_{i=1}^{k} (x - x_i), \qquad x \in I.$$

But Theorem 2 requires that this last sum of integrals be zero if $k \leqslant n$. Thus $k > n$ which implies that the number of zeros of $f_1 - q$ on I exceeds n. A similar argument shows that $f_2 - q$ has more than n zeros on I.

LEMMA 2. *Let* I *be a nondegenerate compact interval of the real line. Let* f_1 *and* f_2 *be real valued measurable functions defined on* I. *Let* $M(x) = \max\{f_1(x), f_2(x)\}$, $x \in I$, *and* $m(x) = \min\{f_1(x), f_2(x)\}$, $x \in I$. *Then for every polynomial* $p \in P$ *we have*

$$\int_I |f_1 - p|\, dx + \int_I |f_2 - p|\, dx = \int_I |m - p|\, dx + \int_I |M - p|\, dx.$$

Proof. The proof follows immediately by noticing that for all $x \in I$ and for all $p \in P$,

$$|f_1(x) - p(x)| + |f_2(x) - p(x)| = |m(x) - p(x)| + |M(x) - p(x)|.$$

The next theorem gives a sufficient condition for the uniqueness of a best approximant to (f_1, f_2).

THEOREM 3. *Let I be a nondegenerate compact interval of the real line. Let f_1 and f_2 be continuous real-valued functions on I. Let q be a polynomial in P such that*

$$\int_I |f_1 - q| \, dx + \int_I |f_2 - q| \, dx = \inf_{p \in P} \left[\int_I |f_1 - p| \, dx + \int_I |f_2 - p| \, dx\right], \tag{2}$$

i.e., q is a best approximant to (f_1, f_2). If there exists $\bar{x} \in I$ such that

$$[f_1(\bar{x}) - q(\bar{x})][(f_2(\bar{x}) - q(\bar{x})] > 0,$$

then q is unique, i.e., if $\bar{q} \in P$ is a best approximant to (f_1, f_2), then $\bar{q} = q$.

Proof. Without loss one may assume, by Lemma 2, that $f_1 \leqslant f_2$. Let q, $\bar{q} \in P$ be best approximants to (f_1, f_2), where there exists $\bar{x} \in I$ such that

$$[f_1(\bar{x}) - q(\bar{x})][f_2(\bar{x}) - q(\bar{x})] > 0.$$

An application of the triangle inequality shows that $q_0 = \frac{1}{2}(q + \bar{q})$ ($\in P$) is also a best approximant to (f_1, f_2). Since $[f_1(x) - q(x)][f_2(x) - q(x)] > 0$ in a neighborhood of $\bar{x}$, it follows, using the equality condition for the triangle inequality, that $|f_1(x) - q(x)| + |f_2(x) - q(x)| > |f_1(x) - f_2(x)|$ in a neighborhood of $\bar{x}$ and hence

$$\int_I |f_1 - q| \, dx + \int_I |f_2 - q| \, dx > \int_I |f_1 - f_2| \, dx.$$

Thus, there exists $\bar{x}_0 \in I$ such that

$$[f_1(\bar{x}_0) - q_0(\bar{x}_0)][f_2(\bar{x}_0) - q_0(\bar{x}_0)] > 0.$$

Further, since q_0, q, $\bar{q}$ are all best approximants, we have

$$\int_I |f_1 - q_0| \, dx + \int_I |f_2 - q_0| \, dx - \frac{1}{2}\left[\int_I |f_1 - q| \, dx + \int_I |f_2 - q| \, dx\right] - \frac{1}{2}\left[\int_I |f_1 - \bar{q}| \, dx + \int_I |f_2 - \bar{q}| \, dx\right] = 0,$$

or

$$\int_I (|f_1 - q_0| - \tfrac{1}{2}|f_1 - q| - \tfrac{1}{2}|f_1 - \bar{q}|) \, dx + \int_I (|f_2 - q_0| - \tfrac{1}{2}|f_2 - q| - \tfrac{1}{2}|f_2 - \bar{q}|) \, dx = 0.$$

Since $|f_i - q_0| \leqslant \frac{1}{2}|f_i - q| + \frac{1}{2}|f_i - \bar{q}|$ on I ($i = 1, 2$), we have

$$|f_i - q_0| - \tfrac{1}{2}|f_i - q| - \tfrac{1}{2}|f_i - \bar{q}| = 0 \text{ on } I \qquad (i = 1, 2).$$

Thus, both $f_i - q$ and $f_i - \bar{q}$ vanish at every zero of $f_i - q_0$ ($i = 1, 2$), i.e., $q = \bar{q}$ at every zero of $f_i - q_0$ ($i = 1, 2$). To complete the proof it suffices to argue that $f_i - q_0$ has more than n zeros for either $i = 1$ or $i = 2$; but this follows immediately from Lemma 1.

The next two lemmas are used in the proof of Theorem 4, which asserts that the best approximant to (f_1, f_2, f_3) is unique.

LEMMA 3. *Let I be a nondegenerate compact interval of the real line. Let f_1, f_2, and f_3 be measurable real valued functions on I. Let* $M(x) = \max\{f_1(x), f_2(x), f_3(x)\}$, $x \in I$, $c(x) = \max[\min\{f_1(x), f_2(x)\}, \min\{f_1(x), f_3(x)\}, \min\{f_2(x), f_3(x)\}]$, $x \in I$, *and* $m(x) = \min\{f_1(x), f_2(x), f_3(x)\}$, $x \in I$. *Then for every polynomial p in P*

$$\begin{aligned}\int_I |f_1 - p| \, dx + \int_I |f_2 - p| \, dx + \int_I |f_3 - p| \, dx \\ = \int_I |m - p| \, dx + \int_I |c - p| \, dx + \int_I |M - p| \, dx.\end{aligned}$$

Proof. The proof follows immediately by noticing that for all $x \in I$

$$\begin{aligned}|f_1(x) - p(x)| + |f_2(x) - p(x)| + |f_3(x) - p(x)| \\ = |m(x) - p(x)| + |c(x) - p(x)| + |M(x) - p(x)|.\end{aligned}$$

LEMMA 4. *Let I be a nondegenerate compact interval of the real line. Let f_1, f_2, and f_3 be real valued continuous functions on I such that $f_1 \leqslant f_2 \leqslant f_3$ on I. Let q be a polynomial in P such that*

$$\begin{aligned}\int_I |f_1 - q| \, dx + \int_I |f_2 - q| \, dx + \int_I |f_3 - q| \, dx \\ = \inf_{p \in P} \left[\int_I |f_1 - p| \, dx + \int_I |f_2 - p| \, dx + \int_I |f_3 - p| \, dx \right].\end{aligned}$$

If $f_i - q$ has at most a finite number of zeros on I ($i = 1, 2, 3$), then $f_2 - q$ has more than n zeros on I.

Proof. Letting $p \equiv 1$ and $m = 3$ in Theorem 2 gives

$$0 \geqslant \left| \int_I \operatorname{sgn}(f_1 - q) \, dx + \int_I \operatorname{sgn}(f_2 - q) \, dx + \int_I \operatorname{sgn}(f_3 - q) \, dx \right|.$$

Thus, $f_2 - q$ has a zero crossing in I. Let $x_1 < x_2 < \cdots < x_k$ be the zero crossings of $f_2 - q$. Since $\operatorname{sgn}(f_2 - q) > 0$ implies that

$$\operatorname{sgn}(f_1 - q) + \operatorname{sgn}(f_2 - q) + \operatorname{sgn}(f_3 - q) > 0 \qquad \text{and} \qquad \operatorname{sgn}(f_2 - q) < 0$$

implies that $\operatorname{sgn}(f_1 - q) + \operatorname{sgn}(f_2 - q) + \operatorname{sgn}(f_3 - q) < 0$, it follows that

$$\left| \int_I \bar{p} \operatorname{sgn}(f_1 - q)\, dx + \int_I \bar{p} \operatorname{sgn}(f_2 - q)\, dx + \int_I \bar{p} \operatorname{sgn}(f_3 - q)\, dx \right| \geqslant \int_I | \bar{p} |\, dx \neq 0,$$

where $\bar{p}(x) = \prod_{i=1}^{k} (x - x_i)$, $x \in I$. But Theorem 2 requires that the sum of integrals above be zero if $k \leqslant n$. Thus $k > n$ which implies that the number of zeros of $f_2 - q$ on I exceeds n.

Remark. The conclusion of the lemma holds without the assumption that $f_1 - q$ and $f_3 - q$ have a finite number of zeros on I. The proof is more involved and is not given here since the weaker form is sufficient for our purposes.

THEOREM 4. *Let I be a nondegenerate compact interval of the real line. Let f_1, f_2 and f_3 be continuous real valued functions on I. Let q be a polynomial in P such that*

$$\int_I | f_1 - q |\, dx + \int_I | f_2 - q |\, dx + \int_I | f_3 - q |\, dx = \inf_{p \in P} \left[\int_I | f_1 - p |\, dx + \int_I | f_2 - p |\, dx + \int_I | f_3 - p |\, dx \right],$$

i.e., q is a best approximant to (f_1, f_2, f_3). Then q is unique, i.e., if $\bar{q} \in P$ is a best approximant to (f_1, f_2, f_3), then $\bar{q} = q$.

Proof. Without loss we may assume, by Lemma 3, that $f_1 \leqslant f_2 \leqslant f_3$ on I. The proof proceeds by contradiction. One assumes that $q, \bar{q} \in P$ are distinct best approximants. An application of the triangle inquality shows that $q_0 = \frac{1}{2}(q + \bar{q})$ ($\in P$) is also a best approximant to (f_1, f_2, f_3). An argument similar to that given in the proof of Theorem 3 shows that $q = \bar{q}$ at every zero of $f_i - q_0$ $(i = 1, 2, 3)$. Since $q \neq \bar{q}$, the number of zeros on I of $f_i - q_0$ $(i = 1, 2, 3)$ is less than n. In particular, the number of zeros of $f_2 - q_0$ is less than n which contradicts the conclusion of Lemma 4.

Theorem 5 gives a sufficient condition for the uniqueness of the best approximant to $(f_1, f_2, \ldots, f_{2m})$. Theorem 6 asserts that the best approximant to $(f_1, f_2, \ldots, f_{2m+1})$ is unique. Their proofs are similar to those of Theorems 3 and 4, respectively.

THEOREM 5. *Let I be a nondegenerate compact interval of the real line. Let $f_1 \leqslant \cdots \leqslant f_{2m}$ be continuous real valued functions on I. Let q be a polynomial in P such that*

$$\sum_{i=1}^{2m} \int_I | f_i - q |\, dx = \inf_{p \in P} \sum_{i=1}^{2m} \int_I | f_i - p |\, dx,$$

i.e., q is a best approximant to $(f_1, \ldots, f_{2m})$. If there exists $\bar{x} \in I$ such that

$$[f_m(\bar{x}) - q(\bar{x})][f_{m+1}(\bar{x}) - q(\bar{x})] > 0,$$

then q is unique, i.e., if $\bar{q} \in P$ is a best approximant to $(f_1, \ldots, f_{2m})$ then $\bar{q} = q$.

Remark. If $f_1, \ldots, f_{2m}$ are continuous real valued functions on I, not necessarily ordered, and

$$h_i(x) = \max_{\pi \in S_i} \min\{f_{\pi(1)}(x), \ldots, f_{\pi(i)}(x)\} \qquad (1 \leqslant i \leqslant 2m),$$

where S_i denotes the set of all one to one mappings of the set of integers $\{1, \ldots, i\}$ into the set of integers $\{1, \ldots, 2m\}$ $(1 \leqslant i \leqslant 2m)$, then we can show that a polynomial which best approximates $\langle f_1, \ldots, f_{2m} \rangle$ is a polynomial which best approximates $(h_1, \ldots, h_{2m})$ and vice versa. Thus, we can give a sufficient condition that the polynomial which best approximates $(f_1, \ldots, f_{2m})$ be unique.

THEOREM 6. *Let I be a nondegenerate compact interval of the real line. Let $f_1, \ldots, f_{2m+1}$ be continuous real valued functions on I. Let q be a polynomial in P such that*

$$\sum_{i=1}^{2m+1} \int_I | f_i - q | \, dx = \inf_{p \in P} \sum_{i=1}^{2m+1} \int_I | f_i - p | \, dx,$$

i.e., q is a best approximant to $(f_1, \ldots, f_{2m+1})$. Then q is unique.

REFERENCES

1. C. B. DUNHAM, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.
2. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart and Winston, New York, 1966.